spherical obstacles[10] and infinitely thin symmetrical obstacles[9] all approach twice the value obtained by geometrical optics. It is found, however, that the back-scattering cross sections of circular cylinders[3] are $\pi/2$ times the geometrical area, those of spheres[11] unity times the geometrical area, and those of thin circular disks square of the geometrical area.

Conclusion

It is concluded from this preliminary investigation that the time-separation or microwave-pulse method of back-scattering measurements can yield accurate results for three-dimensional obstacles of very small scattering cross section and arbitrary shape provided that a judicial choice and design of each component part of the system is made. Thus it supplements the frequency separation method used by Tang[3] for two-dimensional obstacles.

[11] A. Aden, "Electromagnetic Scattering from Metal and Water Spheres," Ph.D. Dissertation, Harvard Univ.; 1950.

Acknowledgment

The author is indebted to Prof. R. W. P. King for his encouragement during the course of the investigation. He also wishes to express his appreciation to Dr. H. Schmitt for his help on numerous occasions.

On Network Representations of Certain Obstacles in Waveguide Regions*

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Summary—**Network representations for a class of obstacles in waveguide regions when the diffraction problem is of a vector type can be obtained by the use of *E*- and *H*-type modes. The special properties of these modes are discussed and highlighted by an example involving the network representation of a periodic strip grating in free space for oblique incidence. Transformations relating the different networks based on various modal representations in rectangular coordinate systems are also discussed.**

I. Introduction

THE problems of the diffraction of electromagnetic waves by obstacles in waveguide or free space are, in general, vector problems. However, in the case of "two-dimensional" obstacles such as the perfectly conducting half plane, infinite periodic gratings, or the infinite circular cylinder in free space, the vector diffraction problem may be decomposed into two independent scalar problems. The same is true in the case of certain structurally similar obstacles in rectangular and parallel plate waveguide. Such decompositions have been employed, for example, by Heins[1] in treating the diffraction of a dipole by a perfectly conducting half plane, and by Levy and Keller[2] in their discussion of diffraction by finitely conducting cylinders at oblique incidence.

In this paper it is shown that modal techniques leading directly to network representations may be employed systematically in the solution of such problems. When the attempt is made to base this approach on the familiar E and H modes propagating perpendicular to the symmetry axis, the desired separation into scalar problems is not possible. On the other hand, the separation into the simpler scalar problems can be effected by appealing to an expansion of the fields in terms of an appropriate alternative set of orthonormal modes. These modes also make it possible to obtain the network representations of problems involving arbitrary angles of incidence directly from the results of the corresponding, strictly two dimensional (incident vector perpendicular to obstacle axis) problems. The matrix relations derived here, which relate the networks based on these modes to networks based on standard E and H modes, further increase the area of applicability of the network solutions.

The modes employed here, which form a complete orthonormal set of vector modes, are designated as the E- and H-type modes. They differ from the familiar H and E modes in that they are characterized by the vanishing of a *transverse*, rather than a longitudinal, field component. To effect the separation into two scalar problems, the modes are chosen such that one sub-set (E-type) has no component of the magnetic field parallel to the axial direction of the "two-dimensional" obstacle, while the second sub-set (H-type) has no com-

* Manuscript received by the PGMTT, July 7, 1958; revised manuscript received, November 4, 1958. The research reported was conducted under Contract AF-19(604)2031, sponsored by the A.F. Cambridge Res. Center, Air Res. and Dev. Command.

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[1] A. E. Heins, "The excitation of a perfectly conducting half-plane by a dipole field," IRE TRANS. ON ANTENNAS AND PROPAGATION, vol. AP-4, pp. 294–296; July, 1956.

[2] B. R. Levy and J. S. Keller, "Diffraction by a Smooth Object," Inst. Math. Sci., New York Univ., N. Y., Res. Rep. EM-109; December, 1957.

ponent of electric field in this direction. In the strictly two-dimensional case, the E- and H-type modes are identical with the H and E modes, respectively.

While the emphasis in this paper is primarily directed towards the application of E- and H-type modes to a technique whereby the network representations for a class of obstacles under general incidence conditions can be obtained, it is necessary to consider the modes themselves in some detail. Modes classified on the basis of vanishing transverse field components have been discussed and employed previously by a number of authors.[3–9]

In Section II the eigenvalue problem for E- and H-type modes in both rectangular and circular cylindrical coordinates is discussed. The connection between the strictly two-dimensional diffraction problem and the general case involving arbitrary angle of incidence is obtained in Section III. Section IV contains an illustration in which the E- and H-type modes are employed to obtain a network representation of a periodic, perfectly conducting strip grating for arbitrary angle of incidence and arbitrary polarization of the incident wave. The linear transformation connecting the various modal representations in rectangular and parallel plate waveguides is treated in Section V. Finally, the E- and H-type mode functions in rectangular coordinates appropriate to free space and to periodic structures in free space are presented in an Appendix.

II. Modal Representations

The total electromagnetic fields in an open or closed waveguide region which possesses an axial direction,[10] here arbitrarily designated as the y direction, can always be represented in terms of two uncoupled scalar functions, each of which satisfies the wave equation[11] when the region is bounded, if at all, by perfect electric or magnetic walls. These scalar functions are essentially the y components of the electric and magnetic fields, E_y and H_y. It is, therefore, suggestive to classify modes on a similar basis (*i.e.*, a sub-set for which $E_y=0$, and one for which $H_y=0$) so that uncoupling of the two modal sub-sets will always result. It must be noted, however, that the above classifications are not sufficient to completely define the mode sets but that a "transmission line direction," *i.e.*, the direction in which the modes are taken to propagate, must also be chosen. It must be noted that the transmission line direction does not necessarily coincide with the axial direction defined above. If the transmission line direction is chosen to coincide with y, then the familiar E and H modes result.[12] If one of the other coordinates is chosen as the transmission line direction, then the resulting modes are E-type ($H_y=0$) and H-type ($E_y=0$) modes. These modes constitute a complete set of vector modes possessing orthogonality properties on surfaces transverse to the transmission line direction.

In the following section, the eigenvalue problems for E- and H-type modes are formulated for waveguide cross sections for which rectangular or polar coordinates are appropriate. The general solutions of these eigenvalue problems are then obtained. Certain explicit mode functions in free space are listed in the Appendix. Mode functions appropriate to parallel plate waveguide, to the conducting wedge, and to periodic structures in free space rotated with respect to the x, y coordinates are available elsewhere.[13]

1. The Eigenvalue Problem in Rectangular Coordinates

Waveguide regions where rectangular coordinates are appropriate are highly degenerate in that three axial directions exist. Here the z direction is arbitrarily chosen as the transmission line direction. The time dependence is taken as exp $j\omega t$.

The vectors transverse (to z) field equations for any uniform waveguide, in the absence of sources, are:[12]

$$\frac{\partial}{\partial z}\boldsymbol{E}_t = -j\omega\mu\left(1_t + \frac{\nabla_t\nabla_t}{k^2}\right)\cdot\boldsymbol{H}_t \times \boldsymbol{z}_0,$$

$$\frac{\partial}{\partial z}\boldsymbol{H}_t = -j\omega\epsilon\left(1_t + \frac{\nabla_t\nabla_t}{k^2}\right)\cdot\boldsymbol{z}_0 \times \boldsymbol{E}_t \qquad (1)$$

where, for rectangular coordinates,

1_t is the transverse unit dyadic $\boldsymbol{x}_0\boldsymbol{x}_0+\boldsymbol{y}_0\boldsymbol{y}_0$,
∇_t is the transverse gradient operator $\boldsymbol{x}_0\partial/\partial x + \boldsymbol{y}_0\partial/\partial y$,

$\boldsymbol{x}_0$, $\boldsymbol{y}_0$ and $\boldsymbol{z}_0$ are unit vectors,

and k is the free space wave number $2\pi/\lambda$.

The desired modal representation of the transverse fields is

[3] N. Marcuvitz, "Waveguide Handbook," Rad. Lab. Ser., vol. 10, McGraw-Hill Book Co., Inc., New York, N. Y., pp. 89–96; 1951.
[4] J. Van Bladel, "Field expandibility in normal modes for a multilayered rectangular or circular waveguide," *J. Franklin Inst.*, vol. 253, pp. 313–321; April, 1952.
[5] C. M. Angulo, "Discontinuities in rectangular waveguide partially filled with dielectric," IRE Trans. on Microwave Theory and Techniques, vol. MTT-5, pp. 68–74; January, 1957.
[6] A. D. Bresler and N. Marcuvitz, "Operator Methods in Electromagnetic Field Theory, chap. 2, Guided Modes in Uniform Cylindrical Waveguide Regions," Microwave Res. Inst., Polytech. Inst. of Brooklyn, N. Y., Rep. No. R-565-57; March, 1957.
[7] R. E. Collin and R. M. Vaillancourt, "Application of Raleigh-Ritz method to dielectric steps in waveguides," IRE Trans. on Microwave Theory and Techniques, vol. MTT-5, pp. 177–184; July, 1957.
[8] L. O. Goldstone and A. A. Oliner, "Leaky Wave Antennas, I: Rectangular Waveguides, II: Circular Waveguides," Microwave Res. Inst., Polytech. Inst. of Brooklyn, N. Y., Reps. No. R-606-57 and R-629-57; August, 1957, and January, 1958.
[9] W. L. Weeks, "Phase Velocities in Rectangular Waveguide Partially Filled with Dielectric," Antenna Lab., Univ. of Illinois, Urbana, Ill., Tech. Rep. No. 28; December, 1957.
[10] Axial direction is defined as a direction such that all cross sections transverse to it are identical in size and shape.
[11] J. A. Stratton, "Electromagnetic Theory," McGraw-Hill Book Co., Inc., New York, N. Y., pp. 350–351; 1941.
[12] N. Marcuvitz, *op. cit.*, Sec. 1.2.
[13] H. M. Altschuler and L. O. Goldstone, "A Class of Alternative Modal Representations for Uniform Waveguide Regions," Microwave Res. Inst., Polytech. Inst. of Brooklyn; Rep. No. R-557-57 February, 1957.

$$\boldsymbol{E}_t(x, y, z) = \sum_i V_i(z)\boldsymbol{e}_i(x, y),$$

$$\boldsymbol{H}_t(x, y, z) = \sum_i I_i(z)\boldsymbol{h}_i(x, y). \qquad (2)$$

Upon substitution of (2) into (1), the transmission line equations and the vector eigenvalue problem for the transverse mode functions [(3) and (4) below] follow readily with the products, $\kappa_i Z_i$ and $\kappa_i Y_i$, playing the role of separation constants. $V_i(z)$ and $I_i(z)$ are hence identified as the modal voltages and currents:

$$\frac{dV_i}{dz} = -j\kappa_i Z_i I_i, \qquad \frac{dI_i}{dz} = -j\kappa_i Y_i V_i, \qquad (3)$$

where κ_i is the modal wave number for propagation along z, and $Z_i = 1/Y_i$ is the modal characteristic impedance. The actual value of Z_i must be chosen appropriately in connection with each particular case. The vector eigenvalue problem for the transverse mode functions is

$$\kappa_i Z_i \boldsymbol{e}_i = \omega\mu\left(1_t + \frac{\nabla_t\nabla_t}{k^2}\right)\cdot \boldsymbol{h}_i \times \boldsymbol{z}_0,$$

$$\kappa_i Y_i \boldsymbol{h}_i = \omega\epsilon\left(1_t + \frac{\nabla_t\nabla_t}{k^2}\right)\cdot \boldsymbol{z}_0 \times \boldsymbol{e}_i. \qquad (4)$$

Eq. (4) may be combined to yield the second order problems for $\boldsymbol{e}_i$ and $\boldsymbol{h}_i$,

$$(\nabla_t^2 + k_{ti}^2)\boldsymbol{e}_i = 0, \qquad (\nabla_t^2 + k_{ti}^2)\boldsymbol{h}_i = 0, \qquad (5)$$

where

$$k_{ti}^2 = k^2 - \kappa_i^2 = k_{xi}^2 + k_{yi}^2.$$

In rectangular coordinates, the preceding equations do not uniquely specify a modal set since the eigenvalue problem posed by (4) is degenerate, in the sense that corresponding to each pair of transverse wave numbers k_{xi}, k_{yi} there are two independent mode functions. These two mode functions may be chosen to be orthogonal to each other in a variety of ways. Each such choice will result in a particular mode set. Two of these sets are of interest here. One is obtained if the familiar condition $\boldsymbol{e}_i = \boldsymbol{h}_i \times \boldsymbol{z}_0$ is imposed. It is comprised of two sub-sets of modes, both associated with the same eigenvalues, namely, the usual E and H modes. These are characterized by vanishing field components in the transmission line direction; in detail, the E modes by $H_z = 0$, and the H modes by $E_z = 0$. If, on the other hand, the condition $e_{yi} = 0$ is imposed, a sub-set of H-type modes results with transverse wave numbers k_{xi} and k_{yi}. The associated modal sub-set (E-type modes) which corresponds to the same transverse wave numbers results upon the imposition of the condition $h_{yi} = 0$. These two modal sub-sets again constitute a complete orthogonal set; the transmission line direction is along z, but the modes are now characterized by vanishing field components along y.

Solutions for the components of the E- and H-type mode functions can be obtained from (5); in particular, it is convenient to fix upon the y components:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_{ti}'^2\right)e_{yi}' = 0, \qquad k_{ti}'^2 = k^2 - \kappa_i'^2 \quad (6a)$$

for the E-type modes, where $h_{yi}' = 0$, and

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_{ti}''^2\right)h_{yi}'' = 0, \qquad k_{ti}''^2 = k^2 - \kappa_i''^2 \quad (6b)$$

for the H-type modes, where $e_{yi}'' = 0$. In order to insure the proper relationship between the components of these transverse vector mode functions, one rewrites the components of (4) in the following forms:

For the E-type modes ($h_{yi}' = 0$),

$$h_{xi}' = -Z_i'\left(\frac{\kappa_i'\omega\epsilon}{k^2 - k_{yi}^2}\right)e_{yi}', \quad e_{xi}' = \frac{1}{k^2 - k_{yi}^2}\frac{\partial^2 e_{yi}'}{\partial x \partial y}. \quad (7)$$

Z_i' may be defined as

$$Z_i' = \frac{k^2 - k_{yi}^2}{\kappa_i'\omega\epsilon}, \qquad (8)$$

so that

$$h_{xi}' = -e_{yi}'. \qquad (9)$$

For the H-type modes ($e_{yi}'' = 0$),

$$e_{xi}'' = Y_i''\left(\frac{\kappa_i''\omega\mu}{k^2 - k_{yi}^2}\right)h_{yi}'', \quad h_{xi}'' = \frac{1}{k^2 - k_{yi}^2}\frac{\partial^2 h_{yi}''}{\partial x \partial y}. \quad (10)$$

Y_i'' may be defined as

$$Y_i'' = \frac{k^2 - k_{yi}^2}{\kappa_i''\omega\mu}, \qquad (11)$$

so that

$$e_{xi}'' = h_{yi}''. \qquad (12)$$

It can be seen from (7) and (10) that these mode functions do not exist when $k^2 = k_{yi}^2$. In such cases an alternative modal description must be employed.

As can be demonstrated, the E-type and H-type mode functions possess the following orthogonality properties:

$$\int_{y_1}^{y_2}\int_{x_1}^{x_2} \boldsymbol{h}_i^\alpha \times \boldsymbol{z}_0 \cdot \boldsymbol{e}_j^{\beta *} dx dy = 0, \; \alpha \neq \beta \text{ and/or } i \neq j, \quad (13)$$

where x_1, x_2, y_1, and y_2 are the appropriate limits of integration, and where both α and β can stand for the prime or the double prime indices; the asterisk stands for complex conjugate. The definition of Z_i' in (8) and Y_i'' in (11) assures that the mode functions are normalized so that

$$\int_{y_1}^{y_2}\int_{x_1}^{x_2} \boldsymbol{h}_i^\alpha \times \boldsymbol{z}_0 \cdot \boldsymbol{e}_j^{\beta *} dx dy = \delta_{ij}\delta_{\alpha\beta} \qquad (14)$$

for bounded[14] regions. It is stressed again that for these modes $\boldsymbol{e}_i^\alpha \neq \boldsymbol{h}_i^\alpha \times \boldsymbol{z}_0$, in contrast to the usual E and H mode case where $\boldsymbol{e}_i^\alpha = \boldsymbol{h}_i^\alpha \times \boldsymbol{z}_0$. In view of this, there is an additional arbitrariness here which has been exploited by defining characteristic impedances as in (8) and (11). The following two scalar orthogonality conditions may be written as a consequence of (14), (9), and (12):

$$\int_{y_1}^{y_2}\int_{x_1}^{x_2} e_{yj}' e_{yi}'^* dx dy = \delta_{ij},$$

$$\int_{y_1}^{y_2}\int_{x_1}^{x_2} h_{yj}'' h_{yi}''^* dx dy = \delta_{ij}. \qquad (15)$$

It is, therefore, apparent that the above choices of characteristic impedance [see (8) and (11)] correspond to a normalization demand on the scalar components of the mode functions e_{yi}' and h_{yi}'' as well.

For the case $k_y = 0$, *i.e.*, when there is no field variation in the y direction, (7)–(12) take on much simpler form. The equations in this form are recognized to be appropriate to the familiar H and E modes; the scalar field components involved satisfy the relation $\boldsymbol{e}_i^\alpha = \boldsymbol{h}_i^\alpha \times \boldsymbol{z}_0$. Eq. (13) now reduces to the usual normalization statement for H and E modes:

$$\int_{x_1}^{x_2} \boldsymbol{e}_i^\alpha \cdot \boldsymbol{e}_j^{\beta *} dx = \delta_{ij}\delta_{\alpha\beta}. \qquad (16)$$

It is seen, then, that when the fields have no variation in the y direction, the E- and H-type mode functions are identically the familiar H and E mode functions (in z), respectively.

The explicit form of the E- and H-type mode functions, of course, depends on the boundary conditions. The actual mode functions for some special cases are presented in the Appendix.

2. *The Eigenvalue Problem for Radial Transmission Line Modes*

The circular cylinder coordinate system appropriate to the following discussion is shown in Fig. 1. The time dependence is again taken as exp $j\omega t$; the radial direction is the transmission line direction. The description employed here is called a radial transmission line description[15] which, it will be shown, is based on a set of E- and H-type modes possessing vector orthogonality properties.

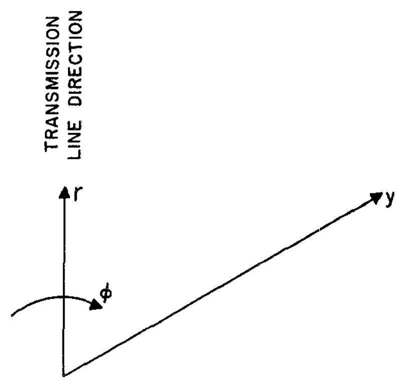

Fig. 1—Circular cylindrical coordinates.

The transverse (to r) field equations in this case are those given in the "Waveguide Handbook."[15] The desired modal representation of the transverse fields is

$$E_y = \sum_i V_i(r) e_{yi}(y, \Phi), \qquad H_y = \sum_i I_i(r) h_{yi}(y, \Phi),$$

$$rE_\Phi = \sum_i V_i(r) e_{\Phi i}(y, \Phi), \quad rH_\Phi = \sum_i I_i(r) h_{\Phi i}(y, \Phi), \quad (17)$$

where, upon substituting (17) into the field equations and applying a separation of variables argument, the modal voltages and currents can be shown to satisfy the radial transmission line equations:

$$\frac{dV_i}{dr} = -j\kappa_i(r) Z_i(r) I_i, \qquad \frac{dI_i}{dr} = -j\kappa_i(r) Y_i(r) V_i. \quad (18)$$

From the field equations, the following two scalar eigenvalue problems are obtained:

$$\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial \Phi^2} + k^2 - k_{yi}'^2 + p_i'^2\right) e_{yi}' = 0$$

$$\text{for } E\text{-type modes: } h_{yi}' = 0, \quad (19)$$

$$\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial \Phi^2} + k^2 - k_{yi}''^2 + p_i''^2\right) h_{yi}'' = 0$$

$$\text{for } H\text{-type modes: } e_{yi}'' = 0. \quad (20)$$

Here k_{yi}', p_i', k_{yi}'', and p_i'' are separation constants. The remaining components of the mode functions are obtained from the field equations and (17) and (18):

$$\left.\begin{aligned} h_{\Phi i}' &= \frac{-\omega\epsilon r}{(k^2 + k_{yi}'^2)} \kappa_i'(r) Y_i'(r) e_{yi}' \quad &(21a)\\ e_{\Phi i}' &= \frac{1}{(k^2 - k_{yi}'^2)} \frac{\partial^2 e_{yi}'}{\partial\Phi\partial y} \quad &(21b)\end{aligned}\right\} \text{for } E\text{-type modes;}$$

$$\left.\begin{aligned} e_{\Phi i}'' &= \frac{\omega\mu r}{(k^2 - k_{yi}''^2)} \kappa_i''(r) Z_i''(r) h_{yi}'' \quad &(22a)\\ h_{\Phi i}'' &= \frac{1}{(k^2 - k_{yi}''^2)} \frac{\partial^2 h_{yi}''}{\partial\Phi\partial y} \quad &(22b)\end{aligned}\right\} \text{for } H\text{-type modes.}$$

In exact analogy with the preceding rectangular case, the following choices are now made:

$$Z_i'(r) = \frac{(k^2 - k_{yi}'^2)}{r\omega\epsilon\kappa_i'(r)}; \qquad Y_i''(r) = \frac{(k^2 - k_{yi}''^2)}{r\omega\mu\kappa_i''(r)}. \quad (23)$$

Eqs. (21a) and (22a) then reduce to

$$h_{\Phi i}' = -e_{yi}'; \qquad e_{\Phi i}'' = h_{yi}''. \quad (24)$$

As before the choices embodied in (23) are equivalent to the following normalization demand:

[14] This and all subsequent orthogonality or normalization statements hold for unbounded regions if δ_{ij} is replaced by $\delta(i-j)$.
[15] N. Marcuvitz, *op. cit.*, sec. 1.7.

$$\int_{y_1}^{y_2}\int_{\Phi_1}^{\Phi_2} e_{yi}' e_{yj}'^* d\Phi dy = \delta_{ij};$$

$$\int_{y_1}^{y_2}\int_{\Phi_1}^{\Phi_2} h_{yi}'' h_{yj}''^* d\Phi dy = \delta_{ij}. \qquad (25)$$

In terms of the scalar functions, with $\boldsymbol{r}_0$, $\boldsymbol{\Phi}_0$, and $\boldsymbol{y}_0$ taken as unit vectors, one defines the vector mode functions as

$$\boldsymbol{e}_i = \boldsymbol{\Phi}_0 e_{\Phi i} + \boldsymbol{y}_0 e_{yi} \quad \text{and} \quad \boldsymbol{h}_i = \boldsymbol{\Phi}_0 h_{\Phi i} + \boldsymbol{y}_0 h_{yi}. \quad (26)$$

Now (23) is equivalent to the following normalization demand on the vector mode functions:

$$\int_{y_1}^{y_2}\int_{\Phi_1}^{\Phi_2} \boldsymbol{h}_j^{\alpha} \times \boldsymbol{r}_0 \cdot \boldsymbol{e}_i^{\beta *} d\Phi dy = \delta_{ij}\delta_{\alpha\beta}. \qquad (27)$$

Eq. (23) expresses Z_i' and Y_i'' in terms of κ_i' and κ_i'', which, it can be shown, are given by

$$[\kappa_i(r)]^2 = (k^2 - k_{yi}^2) - \frac{p_i^2}{r^2}. \qquad (28)$$

These modes exist only when $k^2 \neq k_{yi}^2$. The explicit form of the mode functions in specific cases, of course, depends on boundary conditions.

III. Application to Two-Dimensional Scattering Problems—Arbitrary Angles of Incidence

As has already been pointed out, the total fields in homogeneous waveguide regions uniform in the y direction can be expressed in terms of the scalar field components, E_y and H_y. These components satisfy the scalar wave equation

$$[\nabla_t^2 + (k^2 - k_y^2)]_{H_y}^{E_y} = 0, \qquad (29)$$

where the operator ∇_t^2 is taken as $\nabla^2 - (\partial^2/\partial y^2)$ and the operator $\partial^2/\partial y^2$ as $-k_y^2$. It is apparent that the *functional form* of the solutions of (30) is independent of the value of k_y and that solutions for $k_y \neq 0$ are readily inferred from those for $k_y = 0$. If the solutions for $k_y = 0$ are $E_y = E_y(k)$ and $H_y = H_y(k)$, then those for $k_y \neq 0$ are obtained by replacing k by $\sqrt{k^2 - k_y^2}$ wherever it occurs. This property can be usefully applied, when the field solution of a two-dimensional problem ($k_y = 0$) is known, to obtain a solution for the corresponding problem with $k_y \neq 0$.

It will now be shown that the (E- and H-type) network parameters appropriate to certain two dimensional problems can be similarly modified to yield the network parameters for the case $k_y \neq 0$. This procedure is applicable when the E- and H-type modes are uncoupled both for $k_y = 0$ and $k_y \neq 0$.

Since the y components of the E-type mode functions, e_{yi}', are independent of both k and k_y, the dependence of E_y on $\sqrt{k^2 - k_y^2}$ is associated only with the mode voltages, *i.e.*,

$$E_y(\sqrt{k^2 - k_y^2}) = \sum_i V_i'(\sqrt{k^2 - k_y^2}) e_{yi}'. \qquad (30)$$

The elements of the normalized scattering matrix for a discontinuity which is uniform in y, but otherwise arbitrary, are defined as:

$$\frac{V'_{\text{ref } i}}{V'_{\text{inc } j}} = S_{ij}, \qquad (i, j = 1, 2, 3 \cdots N), \qquad (31)$$

where

$$V_i'(\sqrt{k^2 - k_y^2}) = V'_{\text{inc } i}(\sqrt{k^2 - k_y^2}) + V'_{\text{ref } i}(\sqrt{k^2 - k_y^2}). \qquad (32)$$

From (31) and (32) it may be concluded that

$$S_{ij} = S_{ij}(\sqrt{k^2 - k_y^2}).$$

In view of the dependence of the scattering coefficients on $\sqrt{k^2 - k_y^2}$, it is seen that the scattering matrix for the case $k_y \neq 0$ can be obtained from that for the case $k_y = 0$ by replacing k by $\sqrt{k^2 - k_y^2}$.

When the corresponding impedance matrix Z is normalized, it can be expressed in terms of the scattering matrix. The normalization of the impedance matrix can be accomplished in a variety of ways. The relationship between the scattering matrix and the normalized impedance matrix Z' is

$$Z' = \sqrt{\overrightarrow{Y}_0}(1 + S)(1 - \overrightarrow{Z}_0 \overleftarrow{Y}_0 S)^{-1}\sqrt{\overrightarrow{Z}_0} \qquad (33)$$

where the impedance matrix has been normalized in the following manner:

$$Z' = \sqrt{\overrightarrow{Y}_0} Z \sqrt{\overrightarrow{Y}_0}; \qquad \overleftrightarrow{Z}_0 = (\overleftrightarrow{Y}_0)^{-1}. \qquad (34)$$

Each element of the diagonal matrices $\overrightarrow{Y}_0$ and $\overleftarrow{Y}_0$ is the admittance seen by a mode traveling on an infinite transmission line.[16] The arrows indicate the two directions of travel. Upon examination of (33) one finds that Z', like S, depends only on $\sqrt{k^2 - k_y^2}$. This follows from the fact that the dependence on k (other than that on $\sqrt{k^2 - k_y^2}$) of the admittance matrices $\overrightarrow{Y}_0$ and $\overleftarrow{Y}_0$ is the same, and that this dependence can be factored out as a constant multiplier $F(k)$:

$$\overleftrightarrow{Y}_0 = F(k)\overleftrightarrow{y}_0(\sqrt{k^2 - k_y^2}).$$

Therefore, for the normalized impedance matrix Z', as for the scattering matrix S, the results for $k_y \neq 0$ can be obtained from those for $k_y = 0$ by replacing k by $\sqrt{k^2 - k_y^2}$. Although Z' is normalized in a symmetric manner here ($Z_{ij}' = Z_{ji}'$), the conclusion has been shown to hold for any type of normalization. A similar procedure results in the same conclusions for the scattering and normalized impedance matrices associated with H-type modes.

It is important to recall at this point that the earlier conclusion that E- and H-type modes with $k_y = 0$ are respectively identical to the conventional H and E modes.

[16] For radial and other nonuniform transmission lines $\overrightarrow{Y}_0$ and $\overleftarrow{Y}_0$ are not the characteristic admittances of the transmission lines. An infinite radial line extends from $r = 0$ to $r = \infty$. In the case of uniform transmission lines, $\overrightarrow{Y}_0 = \overleftarrow{Y}_0 =$ characteristic admittance.

One can consequently take advantage of the known equivalent circuits associated with two dimensional ($k_y=0$) E or H mode problems in order to find the networks associated with E- and H-type modes when $k_y \neq 0$. A specific illustration is embodied in the following section.

IV. Illustration of Application of E- and H-Type Modes: Diffraction by an Infinite Strip Grating

The properties of the E- and H-type modes make them particularly convenient for application to a certain class of diffraction problems associated with uniform homogeneous open and closed waveguide regions which may be of unconventional cross section. In more detail, their virtue lies in the fact that these problems involve uncoupled E- and H-type modal sets which can immediately be solved as scalar problems. The solutions of such diffraction problems can be phrased in either field or network terms. They may be exploited to obtain the improper modes of open structures (for given k_y), or the propagation wave numbers k_{y_i} for leaky and closed waveguides, via a transverse resonance procedure.

Let a plane wave

$$\boldsymbol{E} = \boldsymbol{A}e^{-j\boldsymbol{k}.\boldsymbol{r}}, \qquad \boldsymbol{A} = \boldsymbol{x}_0 a_x + \boldsymbol{y}_0 a_y + \boldsymbol{z}_0 a_z$$

be incident at an arbitrary angle upon the doubly infinite strip grating shown in Fig. 2 which is uniform in the y direction. When z is taken as the transmission line direction, the transverse electric field with suppressed y dependence is expressed by the superposition of the two lowest ($m=0$) E-type and H-type mode functions (see Appendix) appropriate to such a periodic structure:

$$\boldsymbol{A}_t e^{-j(k_{x0}x+\kappa_0 z)} = V_0'_{\text{inc}}(z)\boldsymbol{e}_0'(x) + V_0''_{\text{inc}}(z)\boldsymbol{e}_0''(x),$$

where it can be shown that, in terms of the incident wave amplitudes,

$$V_0'_{\text{inc}}(z) = \sqrt{a}\, a_y e^{-j\kappa_0 z},$$

$$V_0''_{\text{inc}}(z) = \sqrt{a}\left(a_x + a_y \frac{k_{x0}k_y}{k^2 - k_y^2}\right) e^{-j\kappa_0 z}.$$

One can therefore regard the problem as two uncoupled scalar problems. For $k_y=0$, these are identically the scalar H mode and E mode problems for which network solutions are known[17] for the E-type mode (*i.e.*, the scalar field e_y) and the H-type mode (*i.e.*, the scalar field h_y) incident, the equivalent circuits are shown in Figs. 3(a) and 3(b), respectively. The representations are at the plane T (taken as $z=0$) in which the strips are located. These solutions are subject to the restriction $a(1-k_{x0}/k)/\lambda < 1$, so that only the lowest E-type mode and H-type mode propagate. The propagation wave number of the associated transmission lines is κ_0; impedances have been normalized to the respective char-

[17] Marcuvitz, *op. cit.*, secs. 5.18 and 5.19.

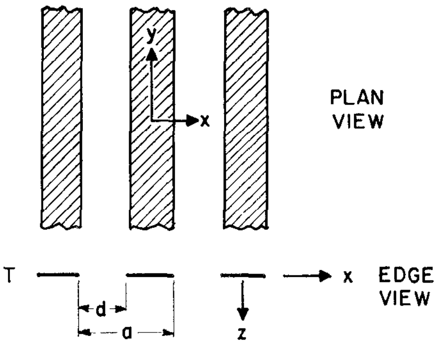

Fig. 2—Doubly infinite strip grating and associated coordinates.

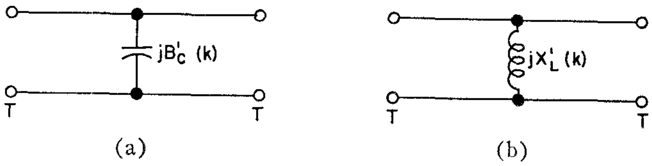

Fig. 3—Equivalent circuits appropriate to the two uncoupled scalar problems when there is no variation in y.

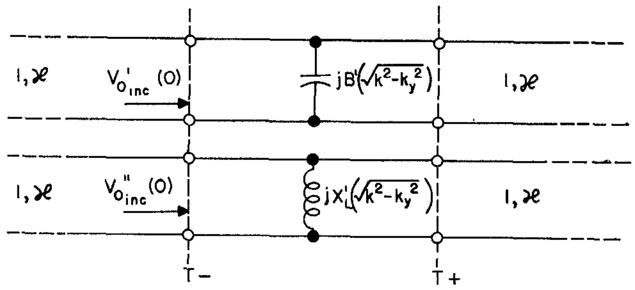

Fig. 4—Equivalent circuit appropriate to the strip grating when there is variation in y.

acteristic impedances. Explicit expressions for the parameters $B_c'(k)$ and $X_L'(k)$ are available[17] in the form

$$B_c'(k) = B_c(a, d, \lambda, \cos\theta)/Y_0'; \qquad Y_0' = \kappa_0/\omega\mu,$$

$$X_L'(k) = X_L(a, d, \lambda, \cos\theta)/Y_0''; \qquad Y_0'' = \omega\epsilon/\kappa_0,$$

where $\cos\theta = \kappa_0/k$, and, of course, $\lambda = 2\pi/k$. The corresponding parameters with $k_y \neq 0$ are $B_c'(\sqrt{k^2-k_y^2})$ and $X_L'(\sqrt{k^2-k_y^2})$. The network solutions with the original plane wave incident then is given by the uncoupled network of Fig. 4, together with the specified incident voltages $V_0'_{\text{inc}}(z)$ and $V_0''_{\text{inc}}(z)$ at $z=0$. The voltage reflection coefficients (with the $\sqrt{k^2-k_y^2}$ dependence understood) at T_- are

$$\Gamma' = -jB_c'/(jB_c' + 2), \qquad \Gamma'' = jX_L'/(jX_L' + 2),$$

from which the voltages at any point z are

$$V_0^\alpha(z) = V_0^\alpha{}_{\text{inc}}(0)(e^{-j\kappa_0 z} + \Gamma^\alpha e^{j\kappa_0 z}), \qquad z < 0,$$

$$V_0^\alpha(z) = V_0^\alpha{}_{\text{inc}}(0)(1 + \Gamma^\alpha)e^{-j\kappa_0 z}, \qquad z > 0,$$

with α standing for prime or double prime. The far field can therefore be written at once as

$$\boldsymbol{E}_t(x, y, z) = [V_0'(z)\boldsymbol{e}_0'(x) + V_0''(z)\boldsymbol{e}_0''(x)]e^{-jk_y y}.$$

V. Relationship Between Modal Representations in Rectangular Coordinates

1. General

In rectangular waveguide regions, which possess more than one axial direction, the eigenvalue problem for the modes is highly degenerate. In such cases a variety of modal representations is possible for the same transmission line direction (z). These sets are not in general independent. In particular, any mode function (corresponding to wave numbers k_{yi} and k_{xi}) of one set can be expressed as a linear combination of two mode functions (corresponding to the same wave numbers) of any one of the other possible sets. In matrix notation[18]

$$\hat{\mathbf{e}}_i = A_i\mathbf{e}_i, \qquad \hat{\mathbf{h}}_i = B_i\mathbf{h}_i, \tag{35}$$

where one defines the vectors

$$\hat{\mathbf{e}}_i \rightarrow \begin{pmatrix}\hat{\mathbf{e}}_i' \\ \hat{\mathbf{e}}_i''\end{pmatrix} \qquad \hat{\mathbf{h}}_i \rightarrow \begin{pmatrix}\hat{\mathbf{h}}_i' \\ \hat{\mathbf{h}}_i''\end{pmatrix},$$

$$\mathbf{e}_i \rightarrow \begin{pmatrix}\mathbf{e}_i' \\ \mathbf{e}_i''\end{pmatrix} \qquad \mathbf{h}_i \rightarrow \begin{pmatrix}\mathbf{h}_i' \\ \mathbf{h}_i''\end{pmatrix}, \tag{36}$$

and the transformation matrices

$$A_i \rightarrow \begin{pmatrix}a_{11}^i & a_{12}^i \\ a_{21}^i & a_{22}^i\end{pmatrix} \qquad B_i \rightarrow \begin{pmatrix}b_{11}^i & b_{12}^i \\ b_{21}^i & b_{22}^i\end{pmatrix}. \tag{37}$$

The orthogonality and normalization of such sets of modes, it is recalled, is given in (13) and (14). By exploiting the orthogonality properties of the two sets of modes in conjunction with (35), the coefficients a_{11}^i, $a_{12}^i \cdots b_{22}^i$ are determined at once. For example,

$$b_{11}^i = \int_{y_1}^{y_2}\int_{x_1}^{x_2} \hat{\mathbf{h}}_i' \; \mathbf{z}_0 \times \mathbf{e}_i'^* dxdy.$$

The relationship between the a and the b coefficients follows upon substitution of the appropriate term from (35), as in

$$b_{11}^i = \int_{y_1}^{y_2}\int_{x_1}^{x_2} \hat{\mathbf{h}}_i' \cdot \mathbf{z}_0 \times (\alpha_{11}\hat{\mathbf{e}}_i' + \alpha_{12}\hat{\mathbf{e}}_i'')^* dxdy = \alpha_{11}^*,$$

where

$$A^{-1} \rightarrow \begin{pmatrix}\alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22}\end{pmatrix}.$$

It follows readily that A_i and B_i satisfy the condition

$$A_i^*\tilde{B}_i = 1. \tag{38}$$

Upon defining the modal voltage and current vectors

$$V_i \rightarrow \begin{pmatrix}V_i' \\ V_i''\end{pmatrix}, \qquad \hat{V}_i \rightarrow \begin{pmatrix}\hat{V}_i' \\ \hat{V}_i''\end{pmatrix},$$

$$I_i \rightarrow \begin{pmatrix}I_i' \\ I_i''\end{pmatrix}, \qquad \hat{I}_i \rightarrow \begin{pmatrix}\hat{I}_i' \\ \hat{I}_i''\end{pmatrix},$$

one can write

$$\tilde{\hat{V}}_i\hat{\mathbf{e}}_i = \tilde{V}_i\mathbf{e}_i, \qquad \tilde{\hat{I}}_i\hat{\mathbf{h}}_i = \tilde{I}_i\mathbf{h}_i, \tag{39}$$

since the same transverse fields must be represented by either set of modes. The tilde sign indicates transposed vectors or matrices. From (39), (13), and (14), the following relations are then obtained:

$$V_i = \tilde{A}_i\hat{V}_i; \qquad I_i = \tilde{B}_i\hat{I}_i. \tag{40}$$

When there is a discontinuity structure in the waveguide region, in which a number of modes are propagating, the voltages and currents at appropriate terminal planes for each mode set are related by appropriate impedance matrices Z and $\hat{Z}$:

$$V = ZI, \qquad \hat{V} = \hat{Z}\hat{I}, \tag{41}$$

where V, $\hat{V}$, I and $\hat{I}$ are the column matrices

$$V \rightarrow (V_i), \qquad \hat{V} \rightarrow (\hat{V}_i), \qquad I \rightarrow (I_i), \qquad \hat{I} \rightarrow (\hat{I}_i).$$

Here the subscript i not only distinguishes the mode voltage and current associated with k_{xi} and k_{yi} but also serves as an index to distinguish two such quantities associated with the same k_{xi} and k_{yi}, when these occur at the input and output terminal planes of the structure. From (40) these voltage and current matrices can be related for the two mode sets as follows:

$$V = \tilde{A}\hat{V}, \qquad I = \tilde{B}\hat{I}, \tag{42}$$

where A and B are diagonal matrices whose elements are the two by two matrices A_i and B_i:

$$A \rightarrow (A_i), \qquad B \rightarrow (B_i),$$

and where

$$A^*\tilde{B} = 1. \tag{43}$$

From (41), (42), and (43) the relationship between the impedance matrices for the two sets of modes then is

$$\hat{Z} = B^*Z\tilde{B} \quad \text{or} \quad Z = \tilde{A}\hat{Z}A^*. \tag{44}$$

Eq. (44) can be represented schematically as shown in Fig. 5, where the transformation which each pair (i) of the mode voltages and currents is subjected to is explicitly exhibited.

The ability to obtain the network description of a discontinuity for one set of modes from that for the other set is useful for certain cases involving more than one discontinuity. For example, if there are two discontinuities in a waveguide region which are far enough apart so that there is no higher mode interaction, the network parameters for each discontinuity may be obtained by using the most convenient set of modes for each. The separate networks may then be combined in a over-all representation in terms of one of the mode sets, via the transformation relations of (44). Such a case is that shown in Fig. 6, where a parallel plate guide contains a slit iris at the plane T_1 and a change in dielectric constant at the plane T_2. Since the E and H modes would

[18] The caret notation serves to distinguish one of the modal sets from the other; prime denotes the E-type sub-set, double prime the H-type sub-set.

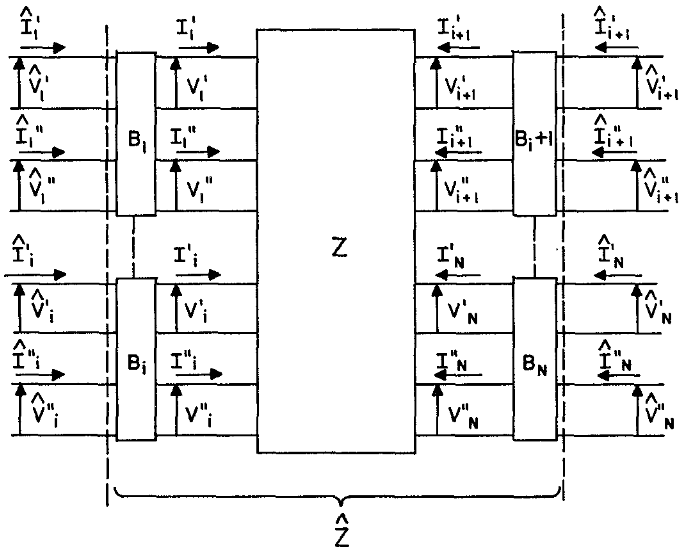

Fig. 5—Schematic representation of the transformation $\hat{Z}=B^*Z\tilde{B}$.

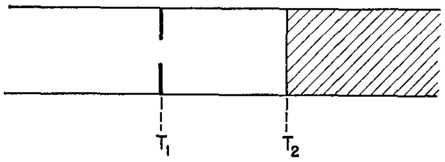

Fig. 6—Parallel plate waveguide containing a slit iris and a change in dielectric constant.

be coupled by the iris but not by the dielectric interface, and vice versa, for the E-type and H-type modes, it is convenient to treat the discontinuity problem at plane T_1 in terms of the E-type and H-type modes, and that at T_2 in terms of E and H modes. Eq. (44) would be employed in combining these results.

VI. Summary

In waveguide regions possessing at least one axial direction the fields can be expanded in terms of E- and H-type modes. In contrast to the usual H and E modes, they are respectively characterized by vanishing magnetic and electric field components in an axial direction (taken as the y direction) transverse to the direction of propagation. They form a complete orthonormal set. The E- and H-type modes, having no periodicity in y (when $k_y=0$), are, respectively, identical to the corresponding H and E modes.

The E- and H- type modes are usefully applied in certain scattering problems in which the obstacles possess uniformity in y, and the modal direction of propagation (or transmission line direction) lies in the plane transverse to y. Under such circumstances the scattering problem is a scalar one in that E- and H-type modes do not couple; under the same set of conditions the familiar E and H modes are coupled, the problem appears to have a vector character, and the usual difficulties associated with vector problems occur.

Assuming that either the field or the network solution of a problem involving incidence normal to the y direction ($k_y=0$) is known, the solution is readily extended to the case with arbitrary angles of incidence ($k_y\neq 0$) by replacing k by $\sqrt{k^2-k_y^2}$ in the scalar fields E_y and H_y of the E- and H-type modes or in the scattering, impedance or admittance parameters of the associated network. This permits the extension to arbitrary angles of incidence of the many known E and H mode solutions of such problems as the half plane, the capacitive slit in parallel plate waveguide, and gratings of various cross sections.

Rectangular waveguide regions (including free space, of course) possess more than one axial direction. In consequence, a variety of modal representations is possible for the same transmission line direction. The relationship between two such representations is given by a transformation matrix which, in turn, can be described in simple network terms. One can consequently combine the networks representing tandem discontinuities even if the emergent transmission lines of each network are defined with respect to different modal representations. This is done by interposing the appropriate transformation matrix or network.

Appendix: E- and H-Type Modes for Free Space (Rectangular Coordinates)

A variety of E- and H-type modal representations is possible for free space. In particular, a representation in terms of either the rectangular or the radial modes can be employed. The modes given below are based on rectangular coordinates.

E-type Modes $h_{yi}'=0$ | *H-type Modes* $e_{yi}''=0$

$$e_{yi}' = -h_{xi}' = \frac{e^{-j(k_{xi}'x+k_{yi}'y)}}{2\pi}, \qquad h_{yi}'' = e_{xi}'' = \frac{e^{-j(k_{xi}''x+k_{yi}''y)}}{2\pi},$$

$$e_{xi}' = \frac{-k_{yi}'k_{xi}'}{k^2-k_{yi}'^2}\,e_{yi}', \qquad h_{xi}'' = \frac{k_{yi}''k_{xi}''}{k^2-k_{yi}''^2}\,h_{yi}'',$$

$$Z_i' = \frac{k^2-k_{yi}'^2}{\kappa_i'\omega\epsilon}, \qquad Z_i'' = \frac{\kappa_i''\omega\mu}{k^2-k_{yi}''^2},$$

$$\kappa_i'^2 = k^2-k_{yi}'^2-k_{xi}'^2, \qquad \kappa_i''^2 = k^2-k_{yi}''^2-k_{xi}''^2,$$

$$-\infty < k_{xi}' < \infty, \qquad -\infty < k_{xi}'' < \infty,$$

$$-\infty < k_{yi}' < \infty; \qquad -\infty < k_{yi}'' < \infty.$$

These modes, when slightly modified, are also appropriate to the consideration of discontinuity structures in free space which possess periodicities along both x and y. The modified modes are then orthogonal in the cell $-a/2<x<a/2$ and $-b/2<y<b/2$. The modification consists of replacing the normalization factor 2π by $\sqrt{ab}$ and recognizing that the wave numbers k_{xi}', k_{xi}'', k_{yi}' and k_{yi}'' take on the discrete values

$$k_{xi}' = k_{xi}'' = \frac{2m\pi}{a}+k_{x0}, \qquad m=0,\pm1,\pm2,\cdots,$$

$$k_{yi}' = k_{yi}'' = \frac{2n\pi}{b}+k_{y0}, \qquad n=0,\pm1,\pm2,\cdots,$$

where k_{x0} and k_{y0} are the x and y components of the propagation wave number of the exciting field, a and b are the periods of the structure in x and y.

In the case of a structure in free space which is periodic (with period a) in x and arbitrary in y, the free space modes are modified by replacing the normalization factor 2π by $\sqrt{2\pi a}$ and recognizing that the wave numbers k_{xi}' and k_{xi}'' take on the discrete values

$$k_{xi}' = k_{xi}'' = \frac{2m\pi}{a} + k_{x0}, \qquad m = 0, \pm 1, \pm 2, \cdots,$$

while the wave numbers k_{yi}' and k_{yi}'' are given by

$$k_{yi}' = k_{yi}'' = \eta + k_{y0}, \qquad -\infty < \eta < \infty.$$

These modes are orthogonal in the strip $-a/2 < x < a/2$ and $-\infty < y < \infty$. In the special case of a structure periodic in x but uniform in y, such as an infinite strip grating, the normalization factor is taken as $\sqrt{a}$, and $\eta = 0$. The exponential y dependence is suppressed.

ACKNOWLEDGMENT

The authors wish to express their appreciation to Dr. N. Marcuvitz and A. D. Bresler for helpful discussions and suggestions in connection with this work.

Reflectors for a Microwave Fabry-Perot Interferometer*

W. CULSHAW†

Summary—**The advantages of microwave interferometers for wavelength and other measurements at millimeter wavelengths are indicated, and a microwave Fabry-Perot interferometer discussed in detail. Analogous to the cavity resonator, this requires reflectors of high reflectivity, small absorption, and adequate size. Stacked dielectric plates, and stacked planar or rod gratings are shown to be suitable forms of reflectors, and equations for the reflectivity, optimum spacing, and bandwidth of such structures are derived. A series of stacked metal plates with regularly spaced holes represents a good design of reflector for very small wavelengths. Fringes and wavelength measurements at 8-mm wavelength are given for one design of interferometer, these being accurate to 1 in 10^4 without any diffraction correction. For larger apertures and reflectors in terms of the wavelength, errors due to diffraction will decrease.**

I. INTRODUCTION

IN conjunction with the efforts directed toward the generation and use of shorter wavelengths in the millimeter region, it is necessary to develop new techniques of transmission and measurement. The facility with which methods based on optical techniques can be used for this purpose improves as the wavelength decreases, in contrast to the conventional waveguide methods, where the dimensions of cavity resonators and other components are in general comparable with the wavelength, with a consequent increase in attenuation and fabrication difficulties. Wavelength measurements can be made with interferometers based on optical principles, and at wavelengths around a few millimeters, such methods would be preferable to the use of a cavity resonator.

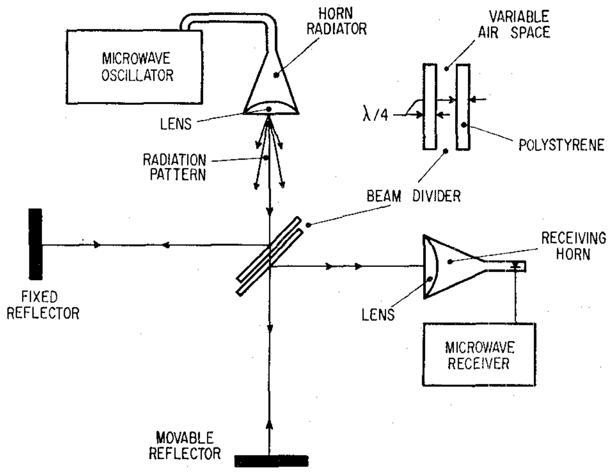

Fig. 1—Microwave form of Michelson interferometer.

A free-space form of Michelson interferometer is shown in Fig. 1; here the beam from the radiating horn is divided by the beam divider into two beams which travel different paths. The two beams then are recombined in the receiving horn, and interference is observed between the two sinusoidal wave trains as one of the reflectors is moved. This interferometer has been operated at $\lambda = 1.25$ cm,[1] the wavelength measurements with a particular form being accurate to a few parts in 10^4 without any correction for diffraction. The free-space beam divider and reference arm can be replaced by a hybrid tee at these wavelengths, and then only a single radiator and reflector are required for the open arm.

* Manuscript received by the PGMTT, July 10, 1958; revised manuscript received, November 10, 1958.
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[1] W. Culshaw, "The Michelson interferometer at millimeter wavelengths," *Proc. Phys. Soc. B*, vol. 63, pp. 939–954; November, 1950.